JOURNAL OF MATERIALS SCIENCE 19 (1984) 2165–2171

Compressive surface strengthening of brittle materials

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A theoretical approach has been put forward for predicting the strengthening of materials by the introduction of surface compressive stresses. An approximate technique was used to determine the closure length of a linear surface crack which extends through the compressive surface layer. The stress intensity factor of the partially closed crack was then determined for the case of an applied tensile stress, with the assumption that the residual surface compressive stress was uniform within the surface layer (step function). The analysis shows that the strengthening depends on the magnitude and depth of the compressive surface stress. It is found that partial crack closure decreases the amount of strengthening compared with that predicted for an open crack, and that for large compressive surface stresses the amount of strengthening can saturate.

1. Introduction

The strength of ceramics or glasses can often be increased by placing their surfaces into compression. Techniques include ion exchange, tempering, glazing, surface chemical reactions and stress-induced phase transformations. Although most of these techniques are well recognized, a theoretical approach to optimization of the strengthening has not been developed. The aim of this paper is to use fracture mechanics to predict the amount of strengthening obtained for a simple residual stress distribution and, in particular, to identify the important material and process parameters that need to be controlled. Such an approach would be expected to be relatively straightforward as many crack loading geometries have been solved. The presence of compressive residual stresses does, however, lead to complications in the analysis, as the crack can be partially closed at the failure condition. These difficulties have impeded the theoretical developments. Partial crack closure in simple configurations has been analysed by several authors [1–5], while for more complex situations, numerical approaches have been used [6–10]. In this paper a simplified approach to the crack closure problem for surface cracks is used and the stress intensity factor is then derived in a more rigorous fashion. The degree of strengthening is then determined by inserting the appropriate fracture criterion, thus identifying the important parameters.

2. Theoretical approach

Consider an infinitely long isotropic plate, width W, which is subjected to the residual stress distribution shown in Fig. 1. This problem is a limiting case of an analysis by Oel and Frechette [11], and it can be shown that the surface (σ_c) and interior (σ_t) stress are given by

$$\sigma_c = \frac{-E\epsilon}{(1-\nu)}\left(\frac{W-2t}{W}\right) \qquad \text{for } t > x > (W-t) \tag{1}$$

and

$$\sigma_t = \frac{2E\epsilon t}{(1-\nu)W} \qquad \text{for } t < x < (W-t) \tag{2}$$

where E is the Young's modulus, ν the Poisson's ratio and ϵ is the linear strain associated with the uniform volume change that occurs at the surface. As can be determined from Equations 1 and 2, a volume increase at the surface leads to a surface compressive stress and a compensating interior tensile stress. Such a redisual stress distribution is expected to be a reasonable approximation for many glazing, enamelling or sealing operations.

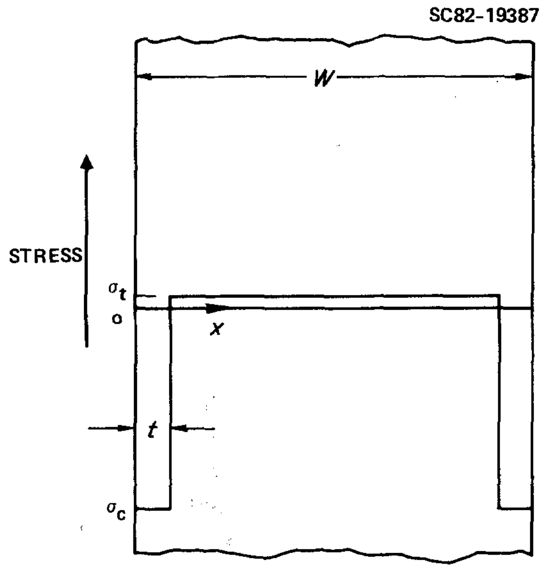

Figure 1 Residual stress distribution in an isotropic flat plate, in which the surface has undergone a uniform volume increase.

For the situations where the surface layers have a different composition from the inside material, Equations 1 and 2 become slightly more complicated [11].

The production of ceramic bodies with a compressive surface layer is expected to lead to strengthening, as a compressive stress will oppose applied tensile stresses, particularly when fracture occurs from flaws at or near the surface. When the entire flaw is subjected to the compressive stress the increase in strength of the body ($\Delta\sigma_f$) will be simply given by

$$\Delta\sigma_f = -\sigma_c \tag{3}$$

In many cases, however, it is expected that the flaw size will be greater than the depth of the compressive zone and it is important to be able to predict the amount of strengthening that will occur. As can be seen from Equation 1, this will lead to an optimization process, as the more shallow the depth of the compressive zone, the larger is the surface compressive stress.

Consider now a semi-infinite body containing a surface crack, length a_0. For this situation the residual stress distribution will be given by

$$\sigma_c \cong \frac{-E\epsilon}{(1-\nu)} \qquad \text{for } x < t \tag{4}$$

and

$$\sigma_t \cong 0 \qquad \text{for } x > t \tag{5}$$

For situations where the surface layers are a different material, the elastic constants in Equation 4 refer to the surface material. In the absence of an applied stress, the residual stress will act to close the crack so that its surfaces are in contact. For example, it is expected that the crack surfaces will be in contact to a depth t, except where $t/a_0 \geqslant 1$, when the crack will be completely closed. When a tensile stress is applied to the body the crack will begin to open until at a critical applied stress the crack surfaces will no longer be in contact. The primary purpose of this paper is to consider situations where $t/a_0 < 1$ and in particular to derive the stress intensity factor (K_I) for this configuration. In this way it will be possible to determine the strengthening in terms of t/a_0. In order to do this however, it is necessary to compute the amount of crack closure as a function of the residual and applied stresses.

2.1. Crack closure analysis

A partially closed surface crack is illustrated in Fig. 2. For the residual stress distribution being considered, the surface crack is assumed to open from its tip back to the surface under the action of an applied stress. This is reasonable when $t/a_0 < 1$, as considered in this analysis. It should, however, be noted that when $t/a_0 \geqslant 1$, the lower constraint at the surface would probably lead to the crack opening in the opposite direction. For

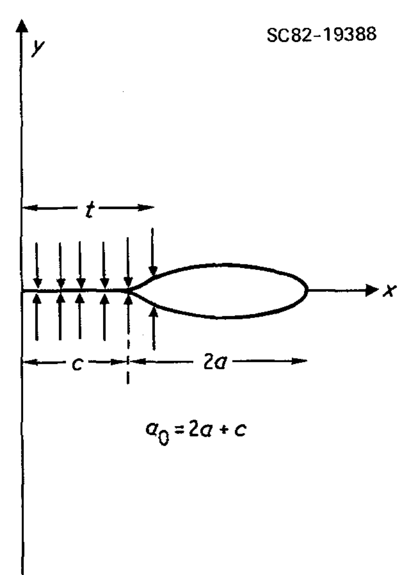

Figure 2 Surface crack in a semi-infinite plate, with partial crack closure due to the surface compression.

these situations, however, the strengthening should be simply given by Equation 3.

It has been noted by other workers [3, 5, 8, 9] that the stress intensity factor at the contact zone (left-hand crack tip in Fig. 2) will be zero. It is possible using this information to determine the crack closure length, c. This procedure is relatively complex, but for this work a simple solution derived by Barenblatt [12] will be used. This solution is strictly only valid for an internal crack in an infinite body but as will be shown later it gives a reasonable description of the closure length. In terms of Fig. 2 the closure distance, c, can be determined from [12]:

$$\frac{a}{a+(t-c)} = \cos\left(\frac{\pi\sigma_a}{2\sigma_c}\right) \tag{6}$$

where σ_a is the applied tensile stress. Using $2a = a_0 - t$, one obtains

$$c_1 \doteq \frac{(1+t_1)\cos\alpha - (1-t_1)}{2\cos\alpha} \tag{7}$$

where $c_1 = c/a_0$, $t_1 = t/a_0$ and $\alpha = \pi\sigma_a/2\sigma_c$. Equation 7 is illustrated in Fig. 3. It can be seen that the crack opens completely at a critical value of σ_a/σ_c which depends on t_1. As indicated earlier, the analysis is not expected to be valid for $t_1 \geqslant 1$. This is reflected in Equation 7, when for $t_1 = 1$, the crack closes completely ($c_1 = 1$). For values of $t_1 \geqslant 1$, it is simply assumed that the crack is completely closed until $\sigma_a = -\sigma_c$ and then it becomes completely open. It should also be noted that the use of Equation 6 ignores the effect of the free surface of the closure length; this is expected to be important as $c_1 \to 0$. For these conditions, however, the crack will be almost completely open and provided the stress intensity factor solution approaches that of an open crack, the approximation should be reasonable.

2.2. Calculation of the stress intensity factor

For an internal crack, length $2a_0$ and closed over its central portion, it has been shown that [2]

$$K_I = 2\left(\frac{a_0}{\pi}\right)^{1/2}(1-c_1^2)^{1/2} \times \int_{c_1}^{1} \frac{\sigma(x_1)x_1\,dx_1}{[(1-x_1^2)(x_1^2-c_1^2)]^{1/2}} \tag{8}$$

where $x_1 = x/a_0$, $\sigma(x_1)$ is the prior stress acting along the plane of the crack, and the crack is closed between $-c$ and c. The coordinate axes are located so the plane of the crack lines along $y = 0$ and the centre of the crack is at $x = 0$. In order to apply this solution to that of a surface crack, the effect of the free surface should be

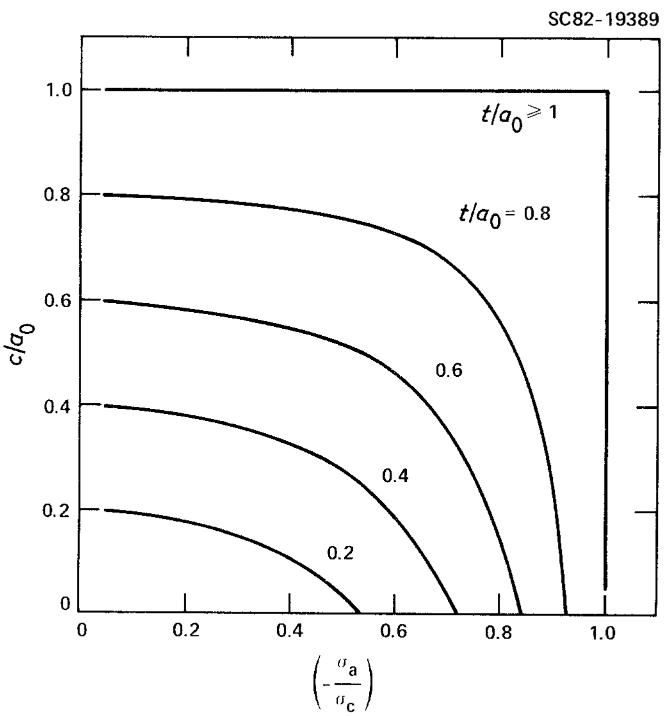

Figure 3 Crack closure length as a function of the compressive zone depth and applied stress.

included. This is usually accomplished by modifying the stress distribution by the factor [13]:

$$f(x_1) = 1.294 - 0.6857x_1^2 + 1.1597x_1^4 - 1.7627x_1^6 + 1.5036x_1^8 - 0.5094x_1^{10} \quad (9)$$

For the residual stress distribution given by Equations 4 and 5 and an applied tensile stress, Equation 8 can be re-written as

$$K_\mathrm{I} = 2\left(\frac{a_0}{\pi}\right)^{1/2}(1-c_1^2)^{1/2} \times \left(\int_{c_1}^{t_1} \frac{(\sigma_\mathrm{a}+\sigma_\mathrm{c})f(x_1)x_1\,\mathrm{d}x_1}{[(1-x_1^2)(x_1^2-c_1^2)]^{1/2}} + \int_{t_1}^{1} \frac{\sigma_\mathrm{a}f(x_1)x_1\,\mathrm{d}x_1}{[(1-x_1^2)(x_1^2-c_1^2)]^{1/2}}\right). \quad (10)$$

Now if we change variables

$$x_1^2 = c_1^2 + (1-c_1^2)u^2 \quad (11)$$

and express $f(x_1)$ in terms of u^2, i.e.

$$f(x_1) = \sum_{k=0}^{5} \alpha_{2k}u^{2k} \quad (12)$$

one obtains

$$K_\mathrm{I} = 2\left(\frac{a_0}{\pi}\right)^{1/2}(1-c_1^2)^{1/2}\sum_{k=0}^{5}\alpha_{2k} \times \left(\int_0^{t_2}\frac{u^{2k}(\sigma_\mathrm{a}+\sigma_\mathrm{c})\,\mathrm{d}u}{(1-u^2)^{1/2}} + \int_{t_2}^{1}\frac{u^{2k}(\sigma_\mathrm{a})\,\mathrm{d}u}{(1-u^2)^{1/2}}\right) \quad (13)$$

where

$$t_2 = \left[\frac{t_1^2-c_1^2}{1-c_1^2}\right]^{1/2} \quad (14)$$

From standard integral tables

$$\int \frac{u^{2k}\mathrm{d}u}{(1-u^2)^{1/2}} = \frac{(2k)!}{(k!)^2}\left[-(1-u^2)^{1/2} \times \sum_{r=1}^{k}\left(\frac{r!(r-1)!u^{2r-1}}{2^{2k-2r+1}(2r)!} + \frac{1}{2^{2k}}\sin^{-1}u\right)\right] \quad (15)$$

The final solution for the stress intensity factor is given by

$$K_\mathrm{I} = \sigma_\mathrm{c}(\pi a_0)^{1/2}(1-c_1^2)^{1/2} \times \left[F_1(t_2) + F_2(c_1)\left(\frac{\sigma_\mathrm{a}}{\sigma_\mathrm{c}} + \frac{2}{\pi}\sin^{-1}t_2\right)\right] \quad (16)$$

where

$$F_1(t_2) = \sum_{k=1}^{5} -\frac{\alpha_{2k}(2k)!(1-t_2^2)^{1/2}\pi}{2(k!)^2} \times \sum_{r=1}^{5}\frac{r!(r-1)!t_2^{2r-1}}{2^{2k-2r+1}(2r)!} \quad (17)$$

and

$$F_2(c_1) = \sum_{k=0}^{5}\frac{\alpha_{2k}(2k)!}{(k!)^2 2^{2k}} \quad (18)$$

It can be shown that Equation 16 agrees with several limiting cases. For example, in the absence of a residual stress ($\sigma_\mathrm{c} = 0$ and $c_1 = 0$)

$$K_\mathrm{I} = 1.1215\,\sigma_\mathrm{a}(\pi a_0)^{1/2} \quad (19)$$

in agreement with the solutions for a surface crack in an applied tensile stress field[13]. For the condition where $-\sigma_\mathrm{a} = \sigma_\mathrm{c}$, $(c_1 = 0)$

$$K_\mathrm{I} = \sigma_\mathrm{a}(\pi a_0)^{1/2}\left(\frac{2}{\pi}\cos^{-1}t_1\right)G(t_1) \quad (20)$$

The solution agrees with that derived for a uniform stress near a crack tip [13], where

$$G(t_1) = 1.1215 - \left[\frac{\pi F_1(t_2)}{2\cos^{-1}(t_1)}\right] \quad (21)$$

Finally, for $\sigma_\mathrm{a} = 0$ and σ_c a tensile stress (i.e. $c_1 = 0$ and $t_2 = t_1$),

$$K_\mathrm{I} = \sigma_\mathrm{c}(\pi a_0)^{1/2}\left(\frac{2}{\pi}\sin^{-1}t_1\right)H(t_1) \quad (22)$$

where

$$H(t_1) = 1.1215 + \frac{\pi F_1(t_2)}{2\sin^{-1}(t_1)} \quad (23)$$

This solution agrees with previous solutions [14, 15] for $0 < t_1 \leqslant 1$, in which a surface crack has a uniform stress near the free surface.

In order to determine the conditions for strengthening, the fracture criterion must be used, i.e. $K_\mathrm{I} = K_\mathrm{c}$. Using this condition and $K_\mathrm{c} = 1.1215\,\sigma_\mathrm{f}^0(\pi a_0)^{1/2}$, where σ_f^0 is the strength of the body in the absence of residual stress, Equation 16 can be rewritten as

$$\frac{\sigma_\mathrm{f}^0}{\sigma_\mathrm{f}} = \frac{0.8917(1-c_1^2)^{1/2}\sigma_\mathrm{c}}{\sigma_\mathrm{f}} \times \left[F_1(t_2) + F_2(c_1)\left(\frac{\sigma_\mathrm{f}}{\sigma_\mathrm{c}} + \frac{2\sin^{-1}t_2}{\pi}\right)\right] \quad (24)$$

Therefore, by choosing $(\sigma_\mathrm{f}/\sigma_\mathrm{c})$, one can calculate c_1 and t_2 and then determine the degree of

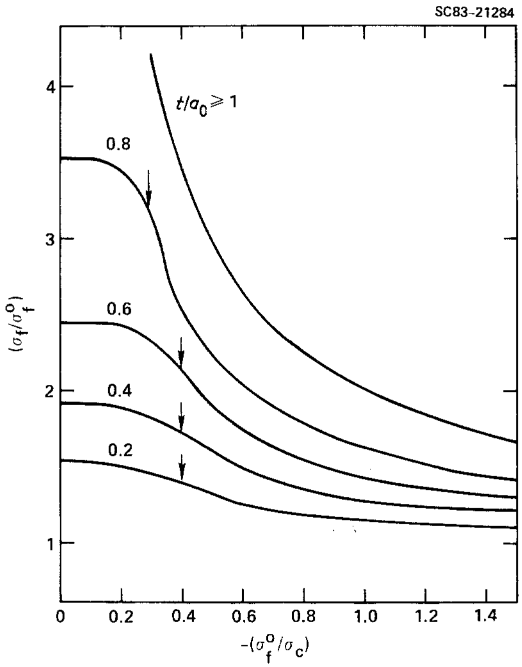

Figure 4 Strengthening due to surface compression in a semi-infinite plate in terms of base strength of material, magnitude of compressive stress and the ratio of compressive zone depth to surface crack size. The arrows in the figure indicate the value of $-\sigma_f^0/\sigma_c$ below which the crack is partially closed.

strengthening (σ_f/σ_f^0). This equation is illustrated in Fig. 4 and will be discussed in the following section. It was noted in the analysis that the partial closure solution simply merged with the open crack solution at a critical value of (σ_f/σ_c). For values of $t_1 \geqslant 1$, the strengthening can be calculated simply using

$$\frac{\sigma_f}{\sigma_f^0} = \left(1 + \frac{\sigma_c}{\sigma_f^0}\right) \tag{25}$$

3. Discussion

The data in Fig. 4 have important consequences for the optimization of compressive strengthening. Maximum strengthening occurs when the crack is completely embedded in the compression zone, i.e. the upper curve in Fig. 4. For these cases, the strengthening depends only on the magnitude of the surface compressive stress, such that increasing the compressive stress increases (σ_f/σ_f^0).

For cases where $(t/a_0) < 1$, the situation becomes more complicated, such that as (t/a_0) decreases, the degree of strengthening decreases. The strengthening can still be increased by increasing the magnitude of the surface compression, but this effect saturates at low values of $-(\sigma_f^0/\sigma_c)$. This reflects the situation where the surface crack is partially closed at failure. For $(\sigma_f^0/\sigma_c) \cong 0$, the crack is so "tightly closed" at the surface that the surface crack now acts more like an internal crack of length $(a_0 - t)/2$, particularly when t_1 approaches unity. This saturation effect can be estimated by comparing the strength of a material containing an internal crack, length $(a_0 - t)/2$, with one containing a surface crack, length a_0, i.e.

$$K_c = 1.1215\,\sigma_f^0(\pi a_0)^{1/2} = \sigma_f[\pi(a_0 - t)/2]^{1/2} \tag{26}$$

or

$$\left(\frac{\sigma_f}{\sigma_f^0}\right)_E = 1.1215\left(\frac{2}{1 - t_1}\right)^{1/2} \tag{27}$$

Table I compares the estimated saturation value of $(\sigma_f/\sigma_f^0)_E$ from Equation 27 with that of Equation 24, and indicates that the estimate is reasonable. For large values of $-(\sigma_f^0/\sigma_c)$, the crack is completely open at failure.

It is worth considering in some more detail the criterion when the surface crack opens completely. From Equation 7 it is straightforward to show that this occurs when

$$\frac{\sigma_f}{\sigma_c} \geqslant \frac{2}{\pi}\left(\frac{1 - t_1}{1 + t_1}\right) \tag{28}$$

This condition is shown by the arrows in Fig. 4. It is interesting to note that this occurs when $\sigma_f^0/\sigma_c \simeq -0.4$. For values greater than this, the amount of strengthening quickly saturates. There is, therefore, no point in increasing the magnitude of the surface compression above $\simeq 3$ to 4 σ_f^0 when $t_1 < 1$. An approach to optimization of compressive surface strengthening is to use compressive surface stresses in this range and then find ways of increasing their depth. Once the surface crack becomes completely embedded in

TABLE I Comparison of saturation values of (σ_f/σ_f^0)

t/a_0	$(\sigma_f/\sigma_f^0)_E$*	(σ_f/σ_f^0)†
0.5	2.243	2.078
0.6	2.508	2.456
0.7	2.896	2.865
0.9	5.016	5.007
0.95	7.093	7.089
0.99	15.860	15.860

*Equation 27.
†Equation 24.

the compressive zone, the maximum benefit from surface compression can be obtained.

It was found in the analysis that crack closure effects reduce the amount of predicted strengthening compared to techniques in which superposition of the residual and applied stress fields is used, such as the work of Swain [16]. It is thus important to determine whether surface cracks are open when failure occurs and simple superposition can be used or whether the more complex partial crack closure analysis is required.

It has been shown that the strengthening depends on the magnitude and depth of the compressive zone. The control of these parameters will depend on the process being used to induce the surface compression. It is important, therefore, to understand the process variables, particularly if σ_c varies with zone depth. Moreover, it should also be remembered that the process itself may change the surface flaw characteristics, for example by subcritical extension of the surface cracks. Such effects can be incorporated into the analysis provided the final crack sizes or their growth rates are known. As an alternative to increasing the depth of the compressive zone, the strengthening can also be increased by reducing the size of the surface cracks.

Further to this point, it has been assumed in the analysis that failure in compressively-strengthened materials still occurs by the extension of surface cracks. However, it clearly is possible that an alternative flaw population, such as internal flaws, could become active. In such cases, the potential strengthening discussed in this paper would not be accomplished and alternative failure models would need to be analysed.

The stress configuration and crack geometry analysed in this paper are somehwat simplistic and there is a need to generalize the effects of more complicated residual stress distributions on more realistic crack shapes. The present analysis does, however, establish that crack closure effects can limit strengthening due to surface compression and indicates approaches for maximizing the degree of strengthening. In addition, there is a need to incorporate the effects of local residual fields. The influence of single indentations in compressively-stressed surfaces has been analysed by Lawn and Marshall [7]. In such cases the influence of crack closure is more complex and bounds are needed to identify the relative interaction between local and long-range surface effects on crack growth.

4. Conclusions

An approach has been put forward for predicting the degree of strengthening of a brittle material, when it is subjected to simple step function residual stress distribution. The approach involves the calculation of the amount of crack closure and the subsequent determination of the stress intensity factor for the particular configuration. It is found that the strengthening simply depends on the magnitude of the compressive stress and the ratio of the compressive layer depth to flaw size (t_1). In particular, when $t_1 \geqslant 1$, the maximum strengthening is obtained. For a given surface compressive stress, the strengthening can be increased by decreasing surface crack size and/or increasing the depth of the compressive zone ($t_1 < 1$). Alternatively, the strengthening can be increased by increasing the magnitude of the compressive stress. This effect, however, saturates at high values of compressive stress due to partial crack closure when $t_1 < 1$. The situation where the crack is partially closed at failure occurs when $-\sigma_f^0/\sigma_c \gtrsim 0.4$. Once this occurs the strengthening quickly saturates. Thus, there is no point in using compressive stresses of higher magnitude and techniques for increasing the depth of the compression zone have a more significant effect on the strengthening.

Acknowledgements

The author would like to acknowledge the financial support of the Office of Naval Research, and discussions with Drs F. F. Lange and W. F. Hall.

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Received 19 July
and accepted 21 September 1983